SOLUTIONS TO EXERCISES

[Last updated 7/7/2015]

Chapter 0.

1. Compute the following matrix products.

   a. $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} (2)(4)+(3)(3) & (2)(6)+(3)(5) \\ (1)(4)+(2)(3) & (1)(6)+(2)(5) \end{pmatrix} =$

   $\begin{pmatrix} 8+9 & 12+15 \\ 4+6 & 6+10 \end{pmatrix} = \begin{pmatrix} \mathbf{17} & \mathbf{27} \\ \mathbf{10} & \mathbf{16} \end{pmatrix}$

   b. $\begin{pmatrix} 3 & 5 & -1 \\ -3 & 7 & 0 \end{pmatrix}\begin{pmatrix} 9 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 27-25-4 \\ -27-35+0 \end{pmatrix} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{-62} \end{pmatrix}$

2. Compute the inverse of each of the following matrices by placing the identity matrix to the right of it and using row reduction.

   a. $\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3/4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3/4 \\ 0 & -1/4 \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ -3/4 & 1 \end{pmatrix} \to$

   $\begin{pmatrix} 1 & 3/4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/4 & 0 \\ 3 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{-2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-4} \end{pmatrix}$

   b. $\begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2/7 & 1/7 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1/7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to$

   $\begin{pmatrix} 1 & 2/7 & 1/7 \\ 0 & 3 & -1 \\ 0 & 34/7 & -11/7 \end{pmatrix}\begin{pmatrix} 1/7 & 0 & 0 \\ 0 & 1 & 0 \\ 3/7 & 0 & 1 \end{pmatrix} \to$

   $\begin{pmatrix} 1 & 2/7 & 1/7 \\ 0 & 1 & -1/3 \\ 0 & 34/7 & -11/7 \end{pmatrix}\begin{pmatrix} 1/7 & 0 & 0 \\ 0 & 1/3 & 0 \\ 3/7 & 0 & 1 \end{pmatrix} \to$

   $\begin{pmatrix} 1 & 0 & 5/21 \\ 0 & 1 & -1/3 \\ 0 & 34/7 & -11/7 \end{pmatrix}\begin{pmatrix} 1/7 & -2/21 & 0 \\ 0 & 1/3 & 0 \\ 3/7 & 0 & 1 \end{pmatrix} \to$

   $\begin{pmatrix} 1 & 0 & 5/21 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1/21 \end{pmatrix}\begin{pmatrix} 1/7 & -2/21 & 0 \\ 0 & 1/3 & 0 \\ 3/7 & -34/21 & 1 \end{pmatrix} \to$

$$\begin{pmatrix}1 & 0 & 5/21\\ 0 & 1 & -1/3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1/7 & -2/21 & 0\\ 0 & 1/3 & 0\\ 9 & -34 & 21\end{pmatrix}\rightarrow$$

$$\begin{pmatrix}1 & 0 & 5/21\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1/7 & -2/21 & 0\\ 3 & -11 & 7\\ 9 & -34 & 21\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{-2} & \mathbf{8} & \mathbf{-5}\\ \mathbf{3} & \mathbf{-11} & \mathbf{7}\\ \mathbf{9} & \mathbf{-34} & \mathbf{21}\end{pmatrix}$$

3. Compute the determinant of each of the following matrices.
   a. $\det\begin{pmatrix}6 & 9\\ 1 & -1\end{pmatrix} = (6)(-1) - (9)(1) = \mathbf{-15}$
   b. $\det\begin{pmatrix}5 & 3\\ 7 & 11\end{pmatrix} = (5)(11) - (3)(7) = \mathbf{34}$
4. $\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}^{-1} = \frac{\mathbf{1}}{\boldsymbol{\alpha\delta-\beta\gamma}}\begin{bmatrix}\boldsymbol{\delta} & \boldsymbol{-\beta}\\ \boldsymbol{-\gamma} & \boldsymbol{\alpha}\end{bmatrix}; \begin{bmatrix}7 & 3\\ 2 & 1\end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix}1 & -3\\ -2 & 7\end{bmatrix} = \begin{bmatrix}\mathbf{1} & \mathbf{-3}\\ \mathbf{-2} & \mathbf{7}\end{bmatrix}$
5. Elementary row operations do not change the solutions to the system of equations, so Gaussian elimination can be used to simplify a system of equations. It transforms a matrix into row echelon form, which makes it easy to see what the solutions are.
6. Yes; it is possible to get the final solution simply by this operation, but it is more convenient to use all three elementary row operations to solve simultaneous equations.
7. Solve the following systems of linear equations using the Gauss-Jordan Method.
   a. We write the system of linear equations out as an augmented matrix. Then, we solve using the Gauss-Jordan Method.

$$\begin{pmatrix}1 & 1 & 1 & 5\\ 2 & 3 & 5 & 8\\ 4 & 0 & 5 & 2\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & 1 & 5\\ 0 & 1 & 3 & -2\\ 4 & 0 & 5 & 2\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & 1 & 5\\ 0 & 1 & 3 & -2\\ 0 & -4 & 1 & -18\end{pmatrix}\rightarrow$$

$$\begin{pmatrix}1 & 0 & -2 & 7\\ 0 & 1 & 3 & -2\\ 0 & -4 & 1 & -18\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & -2 & 7\\ 0 & 1 & 3 & -2\\ 0 & 0 & 13 & -26\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & -2 & 7\\ 0 & 1 & 3 & -2\\ 0 & 0 & 1 & -2\end{pmatrix}\rightarrow$$

$$\begin{pmatrix}1 & 0 & 0 & 3\\ 0 & 1 & 3 & -2\\ 0 & 0 & 1 & -2\end{pmatrix}\rightarrow\begin{pmatrix}1 & 0 & 0 & 3\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & -2\end{pmatrix}.$$

   We obtain the solution $\boldsymbol{x = 3, y = 4, z = -2}$.
   b. We write the system of linear equations out as an augmented matrix. Then, we solve using the Gauss-Jordan Method.

$$\begin{pmatrix} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 3/2 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 3 & -1 & 3/2 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 3/2 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & -4 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 3/2 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We obtain the solution $\boldsymbol{x} = \frac{\boldsymbol{7z}}{\boldsymbol{4}}, \boldsymbol{y} = \frac{\boldsymbol{1}}{\boldsymbol{2}} - \frac{\boldsymbol{z}}{\boldsymbol{4}}, \boldsymbol{z} = \boldsymbol{z}$, where z is a free variable.

c. We write the system of linear equations out as an augmented matrix. Then, we solve using the Gauss-Jordan Method.

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 2 & -1 & 2 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & 0 & -1/3 & -8/3 & 17/3 \\ 0 & -3 & -2 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & 0 & -1/3 & -8/3 & 17/3 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

We obtain the solution $\boldsymbol{a = 1, b = 2, c = -1, d = -2}$.

8. You multiply by the inverse of $\boldsymbol{B}$, namely, $\boldsymbol{B^{-1}}$.
9. Suppose that $\boldsymbol{A}$ is a 10×100 matrix, $\boldsymbol{B}$ is a 100×5 matrix, $\boldsymbol{C}$ is a 5×50 matrix, and $\boldsymbol{D}$ is a 50×1 matrix. Compute the number of multiplications needed in order to compute each of the following. Then, state which order of multiplication you found to be optimal.
   a. ((AB)C)D → $5000 + 2500 + 500 = \mathbf{8000}$
   b. (AB)(CD) → $5000 + 250 + 50 = \mathbf{5300}$
   c. (A(BC))D → $2500 + 50000 + 500 = \mathbf{53000}$
   d. A((BC)D) → $2500 + 5000 + 1000 = \mathbf{8500}$
   e. A(B(CD)) → $250 + 500 + 1000 = \mathbf{1750}$

   The last order of multiplication, A(B(CD)), was optimal.